THE TOPOLOGY AT INFINITY OF A MANIFOLD SUPPORTING AN $L^{q,p}$-SOBOLEV INEQUALITY

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Abstract. This note aims to prove that a complete manifold supporting a general $L^{q,p}$-Sobolev inequality is connected at infinity provided the negative part of its Ricci tensor is small (in a suitable spectral sense). In the route, we deduce potential theoretic and volume properties of the ends of a manifold enjoying the $L^{q,p}$-Sobolev inequality. Our results are related to previous work by I. Holopainen, [10], and S.W. Kim, and Y.H. Lee, [12].

Introduction

In this paper we discuss the topology at infinity of a complete non compact Riemannian manifold $(M, \langle , \rangle)$ supporting a general $L^{q,p}$-Sobolev inequality of the type

$$S_{q,p} \|\varphi\|_{L^q} \leq \|\nabla\varphi\|_{L^p}, \tag{1}$$

for some constant $S_{q,p} > 0$ and for every $\varphi \in C_c^\infty(M)$. It is known from a seminal work by Pansù, [15], and the recent extensions in [7], that the validity of (1) is related to a "global" cohomology theory which is sensitive only on the geometry at infinity of the underlying manifold, the so called $L^{q,p}$-cohomology, and gives information on the solvability of non-linear differential equations involving the p-Laplace operator (on differential forms). For instance, using this circle of ideas one obtains that if the first $L^{q,p}$-cohomology group vanishes, then the existence of a non-constant p-harmonic function $v : M \to \mathbb{R}$ with finite p-energy $|\nabla v| \in L^p(M)$ implies that M is not simply connected. In this paper, we use p-harmonic function techniques to show that, under the validity of (1), and assuming a suitable control on the Ricci tensor, the underlying manifold M is connected at infinity. This means that, for every arbitrarily large compact set $K \subset M$, the open set $M \backslash K$ has only one unbounded connected component, namely, M has only one end. It is a well known consequence of the splitting theorem by J. Cheeger and D. Gromoll, [3], that a complete manifold with non-negative Ricci curvature has at most two ends. Furthermore, if the Ricci curvature is positive at some point, then we have connectedness at infinity. With the aid of the $L^{q,p}$-Sobolev inequality, and extending in a nontrivial way to the case $p \neq 2$ previous results in [4], [14], [16], we will conclude the connectedness

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at infinity even in the presence of some amount of negative Ricci curvature. More precisely, we shall prove the following result. Recall that the bottom of the spectrum of the Schrodinger operator $L = \Delta + Hq(x)$ is given by

$$\lambda_1^{-L}(M) = \inf_{\varphi \in C_c^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla\varphi|^2 - \int_M Hq(x)\varphi^2}{\int_M \varphi^2}.$$

Theorem 1. *Let $(M, \langle , \rangle)$ be a complete non compact Riemannian manifold of dimension $\dim M = m$. Let $q > p \geq 2$ be such that*

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{m},$$

and assume that M supports an $L^{q,p}$-Sobolev inequality of the type

$$S_{q,p} \|\varphi\|_{L^q} \leq \|\nabla\varphi\|_{L^p},$$

for some constant $S_{q,p} > 0$ and for every $\varphi \in C_c^\infty(M)$. Assume that the Ricci tensor of M is such that

$$^M Ric \geq -q(x) \text{ on } M$$

for a suitable function $q \in C(M)$. If the Schrodinger operator $L = \Delta + Hq(x)$ satisfies

$$\lambda_1^{-L}(M) \geq 0, \tag{2}$$

for some constant $H > p^2/4(p-1)$, then, M has only one end.

As we briefly mentioned above, the validity of an $L^{q,p}$-Soblev inequality gives information on the $L^{q,p}$-cohomology of the underlying manifold. We note that, from this point of view, the main theorem can be restated in the following equivalent form. We refer the reader e.g. to the paper [7] for the relevant definitions.

Theorem 2. *Let $(M, \langle , \rangle)$ be a complete, m-dimensional Riemannian manifold satisfying the following requirements:*

(a) M has at least two ends;
(b) $^M Ric \geq -q(x)$, for a suitable function $q \in C^0(M)$ such that

$$\lambda_1^{-\Delta - Hq(x)}(M) \geq 0,$$

and for some constant $H > p^2/4(p-1)$, $p \geq 2$.

Then, for every $1/p - 1/q \leq 1/m$, the first $L^{q,p}$-cohomology space of M is unreduced.

1. Preliminary results from non-linear potential theory

For the sake of completeness, we first recall some definitions and prove some facts from non-linear potential theory; see [9, 20].

Definition 3. *A Riemannian manifold $(M, \langle,\rangle)$ is said to be p-parabolic if for some compact set of positive volume $K \subset M$, we have*

$$\mathrm{cap}_p (K) := \inf \int_M |\nabla \varphi|^p = 0,$$

where the infimum is taken with respect to all functions $\varphi \in C_c^\infty (M)$ such that $\varphi \geq 1$ on K. Otherwise, we say that M is p-hyperbolic.

It is not hard to prove that on a compact manifold, the p-capacity of any compact set vanishes. The next result gives further equivalent characterizations of p-parabolicity. Note that the equivalence between (ii) – (iv) below is proved following arguments valid in the case $p = 2$ (see e.g. [16]) while the equivalence with condition (v) is a result in [6]. Furthermore, the equivalence (i) – (ii) was already observed in [9] using the non-linear Green function introduced by the author. However, we shall provide a new and somewhat direct argument that we feel interesting in its own. Finally, to the best of our knowledge, the explicit equivalence (iii) – (iv) has never been observed before.

Theorem 4. *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold. The following conditions are equivalent.*

(i) *M is p-parabolic.*
(ii) *If $u \in C(M) \cap W^{1,p}_{loc}(M)$ is a bounded above weak solution of $\Delta_p u \geq 0$ then u is constant.*
(iii) *There exists a relatively compact domain D in M such that, for every function $\varphi \in C(M \setminus D) \cap W^{1,p}_{loc}(M \setminus \overline{D})$ which is bounded above and satisfies $\Delta_p \varphi \geq 0$ weakly in $M \setminus \overline{D}$, $\sup_{M\setminus D} \varphi = \max_{\partial D} \varphi$.*
(iv) *For every domain $\Omega \subset M$ and for every $\psi \in C\left(\bar{\Omega}\right) \cap W^{1,p}_{loc}(\Omega)$ which is bounded above and satisfies $\Delta_p \psi \geq 0$ weakly on Ω, $\sup_\Omega \psi = \sup_{\partial\Omega} \psi$.*
(v) *There exists a compact set $K \subset M$ with the following property. For every constant $C > 0$, there exists a compactly supported function $v \in W^{1,p}(M) \cap C(M)$ such that*

$$\|v\|_{L^p(K)} > C \|\nabla v\|_{L^p(M)} .$$

Proof. (i) $\Rightarrow$ (ii). Let M be p-parabolic, so that, for every compact set K, $\mathrm{cap}_p(K) = 0$, and assume by contradiction that there exists a positive, p-superharmonic function u. By translating and scaling, we may assume that $\sup u > 1$ and $\inf u = 0$. Note that, by the strong maximum principle (see, e.g., [11] Theorem 7.12) u is strictly positive on M. Next let D be a relatively compact domain with smooth boundary contained in the superlevel set $\{u > 1\}$ and let D_i be an exhaustion of M consisting of relatively compact domains with smooth boundary such that $\overline{D} \subset\subset D_1$, and for every i let h_i be the

solution of the the Dirichlet problem

$$\begin{cases} \Delta_p h_i = 0, \text{ on } D_i \setminus D \\ h_i = 1, \text{ on } \partial D \\ h_i = 0, \text{ on } \partial D_i. \end{cases}$$

By a result of Tolksdorf [19], $h_i \in C^{1,\alpha}_{loc}\left(D_i \setminus \overline{D}\right)$. Furthermore, since D and D_i have smooth boundaries, applying Theorem 6.27 in [11] with θ any smooth extension of the piecewise function

$$\theta_0 = \begin{cases} 1, \text{ on } \partial D \\ 0, \text{ on } \partial D_i, \end{cases}$$

we deduce that h_i is continuous on $\overline{D}_i \setminus D$. By the strong maximum principle $0 < h_i < 1$ in $D_i \setminus \overline{D}$ and $\{h_i\}$ is increasing, so that, by the Harnack principle, $\{h_i\}$ converges locally uniformly on $M \setminus D$ a function h which is continuous on $M \setminus D$, p-harmonic on $M \setminus \overline{D}$ and satisfies $0 < h \leq 1$ on $M \setminus \overline{D}$ and $h = 1$ on ∂D. Again, $h \in C\left(M \setminus D\right) \cap C^{1,\alpha}_{loc}\left(M \setminus \bar{D}\right)$.

Moreover, since h_i is the p-equilibrium potential of the condenser $(\overline{D}, D_i)$,

$$\operatorname{cap}_p\left(\overline{D}, D_i\right) = \int |\nabla h_i|^p = \inf \int |\nabla \varphi|^p,$$

where the infimum is taken with respect to $\varphi \in C^\infty_c\left(D_i\right)$ such that $\varphi = 1$ on ∂D. Think of each h_i extended to be zero off D_i. Therefore $\left\{\int_{M \setminus \overline{D}} |\nabla h_i|^p\right\}$ is decreasing and the sequence $\{h_i\} \subset W^{1,p}\left(\Omega\right)$ is bounded on every compact domain Ω of $M \setminus \overline{D}$. By the weak compactness theorem, see, e.g., Theorem 1.32 in [11], $h \in W^{1,p}\left(\Omega\right)$, and $\nabla h_i \to \nabla h$ weakly in $L^p\left(\Omega\right)$. In particular,

$$\int_\Omega |\nabla h|^p \leq \liminf_{i \to +\infty} \int_{D_i \setminus D} |\nabla h_i|^p.$$

On the other hand, it follows easily from the definition of capacity, that $\lim_i \operatorname{cap}_p(\overline{D}, D_i) = \operatorname{cap}_p(\overline{D}) = 0$. Thus, letting $\Omega \nearrow M \setminus \overline{D}$ we conclude that

$$\int_{M \setminus \overline{D}} |\nabla h|^p = 0,$$

so that h is constant, and since $h = 1$ on ∂D, $h \equiv 1$. Finally, since u is p-superharmonic and $u > h_i$ on $\partial D \cup \partial D_i$, by the comparison principle, $u \geq h_i$ on $D_i \setminus D$, and letting $i \to \infty$ we conclude that $u \geq 1$ on M, contradiction.

(ii) $\Rightarrow$ (i). Given a relatively compact domain D, let h_i and h be the functions constructed above, and extend h to be 0 in D, so that h is continuous on M, bounded, and satisfies $\Delta_p h \geq 0$ weakly on M. Thus (ii) implies that h is identically equal to 1. On the other hand, since the functions h_i belong to $W^{1,p}_0(M)$, Lemma 1.33 in [11] shows that ∇h_i converges to ∇h weakly in $L^p(M)$. By Mazur's Lemma (see Lemma 1.29 in [11]) there exists a sequence v_k of convex combinations of the h_i's such that ∇v_k converges to ∇h strongly in L^p. Thus v_k is continuous, compactly supported, identically

equal to 1 on $\overline{D}$ (because so are all the h_i's) and $\int_M |\nabla v_k|^p \to \int |\nabla h|^p = 0$, showing that $\mathrm{cap}_p(\overline{D}) = 0$, and M is p-parabolic.

(iii) $\Rightarrow$ (iv). Assume that (iii) holds, and suppose by contradiction that there exist a domain Ω and a function ψ as in (iv) for which $\sup_{\partial\Omega}\psi < \sup_\Omega \psi$. Note that, by the strong maximum principle, Ω is unbounded. Choose $0 < \varepsilon < \sup_\Omega \psi - \sup_{\partial\Omega}\psi$ sufficiently near to $\sup_\Omega \psi - \sup_{\partial\Omega}\psi$ so that $\overline{D} \cap \{\psi > \sup_{\partial\Omega}\psi + \varepsilon\} = \emptyset$. This is possible according to the strong maximum principle, because $\overline{D}$ is compact. Define $\tilde{\psi} \in C(M) \cap W^{1,p}_{loc}(M)$ by setting

$$\tilde{\psi}(x) = \max\{\sup_{\partial\Omega}\psi + \varepsilon, \psi(x)\}$$

and note that $\Delta_p \tilde{\psi} \geq 0$ on M. According to property (iii),

$$\max_{\partial D} \tilde{\psi} = \sup_{M\setminus D} \tilde{\psi}.$$

However, since $\overline{D} \cap \{\psi > \sup_{\partial\Omega}\psi + \varepsilon\} = \emptyset$,

$$\max_{\partial D} \tilde{\psi} = \sup_{\partial\Omega}\psi + \varepsilon < \sup_{\Omega}\psi,$$

while

$$\sup_{M\setminus D} \tilde{\psi} = \sup_{\Omega}\psi.$$

The contradiction completes the proof.

(iv)$\Rightarrow$(iii). Trivial.

(iv)$\Rightarrow$(ii). Assume by contradiction that there exists $u \in C(M) \cap W^{1,p}_{loc}(M)$ which is non-constant, bounded above and satisfies $\Delta_p u \geq 0$ weakly on M. Given $\gamma < \sup u$, the set $\Omega_\gamma = \{u > \gamma\}$ is open, and u is continuous and bounded above in $\overline{\Omega}_\gamma$, satisfies $\Delta_p u \geq 0$ weakly in Ω_γ and $\max_{\partial\Omega_\gamma} u < sup_{\Omega_\gamma} u$, contradicting (iv).

(ii)$\Rightarrow$(iv). If there exists $\psi \in C\left(\overline{\Omega}\right) \cap W^{1,p}_{loc}(\Omega)$ satisfing $\Delta_p \psi \geq 0$ and $\sup_\Omega \psi > \max_{\partial\Omega}\psi + 2\varepsilon$, for some $\varepsilon > 0$, then

$$\psi_\varepsilon = \begin{cases} \max\{\psi, \max_{\partial\Omega}\psi + \varepsilon\} & \text{in } \Omega \\ \max_{\partial\Omega} + \varepsilon & \text{in } M\setminus\Omega, \end{cases}$$

is a non-constant, bounded above, weak solution of $\Delta_p \psi_\varepsilon \geq 0$ on M. This contradicts (ii).

For the equivalence (i) $\Leftrightarrow$ (v), see [6] Theorem 3.1. □

We now localize the concept of parabolicity on a given end. Recall that, by definition, an end E of M with respect to a compact domain K is any of the unbounded connected components of $M\setminus K$.

Definition 5. *An end E of the Riemannian manifold $(M, \langle,\rangle)$ is said to be p-parabolic if, for every compact set $K \subset \bar{E}$,*

$$\mathrm{cap}_p(K, E) = \inf \int_E |\nabla\varphi|^p = 0,$$

where the infimum is taken with respect to all $\varphi \in C_c^\infty\left(\bar{E}\right)$ *such that* $\varphi \geq 1$ *on* K*.*

We have the following characterizations of the parabolicity of ends. Recall that the Riemannian double of a manifold E with smooth, compact boundary ∂E is a smooth Riemannian manifold (without boundary) $\mathcal{D}(E)$ such that (i) $\mathcal{D}(E)$ is complete (ii) $\mathcal{D}(E)$ is homeomorphic to the topological double of E and (iii) there is a compact set $K \subset \mathcal{D}(E)$ such that $\mathcal{D}(E) \setminus K$ has two connected components, both isometric to E. Observe that this is not uniquely defined, but all such "doubles"are bilipshitz equivalent.

Theorem 6. *An end* E *with smooth boundary* ∂E *is said to be* p*-parabolic if either of the following equivalent conditions is satisfied:*

(i) *For every continuous* $\phi : \bar{E} \to \mathbb{R}$ *which is bounded above and* p*-subharmonic,* $\sup_E \phi = \max_{\partial E} \phi$*.*
(ii) *The (Riemannian) double* $\mathcal{D}(E)$ *of* E *is a* p*-parabolic manifold without boundary.*

Condition (i) in Theorem 6 yields easily the following necessary and sufficient condition for an end to be p-hyperbolic.

Corollary 7. *An end* E *is* p*-hyperbolic if and only if there exists a function* $\psi \in C(\overline{E}) \cap W^{1,p}_{loc}(E)$ *which is* p*-superharmonic and such that* $\inf_E \psi = 0$ *and* $\psi \geq 1$ *on* ∂E*.*

Corollary 7 allows us to obtain the existence of special p-harmonic functions on p-hyperbolic ends (whose existence, in view of Theorem 4 in fact characterizes p-hyperbolic ends).

Lemma 8. *Let* E *be a* p*-hyperbolic end of* $(M, \langle , \rangle)$ *with smooth boundary. Then, there exists a non-constant* p*-harmonic function* $h \in C\left(\bar{E}\right) \cap C^{1,\alpha}_{loc}(E)$ *such that:*

(1) $0 < h \leq 1$ *in* $\bar{E}$*,*
(2) $h = 1$ *on* ∂E*,*
(3) $\inf_{\bar{E}} h = 0$*,*
(4) $|\nabla h| \in L^p\left(\bar{E}\right)$*.*

Proof. Take a smooth exhaustion D_i of M with $\partial E \subset D_0$. Set $E_i = E \cap D_i$ and solve the Dirichlet problem

$$\begin{cases} \Delta_p h_i = 0, & \text{on } E_i \\ h_i = 1, & \text{on } \partial E \\ h_i = 0, & \text{on } \partial D_i \cap E. \end{cases}$$

By the arguments used in the proof of Theorem 4, $h_i \in C^{1,\alpha}_{loc}(E_i) \cap C(\overline{E}_i)$, $0 < h_i < 1$ in E_i, it is increasing and converges (locally uniformly) to a p-harmonic function h on $h \in C\left(\bar{E}\right) \cap C^{1,\alpha}_{loc}(E)$ satisfying $0 < h \leq 1$ and $h = 1$ on ∂E. Since E is p-hyperbolic, there exists a function ψ with the properties listed in Corollary 7. By the comparison principle, $h_i \leq \psi$ for

every i, and passing to the limit, $h \le \psi$, so that $\inf_E h = 0$ and in particular h is non-constant.

To prove that h has finite p-energy we argue as in the proof of Theorem 4, to show that $\left\{\int_E |\nabla h_i|^p\right\}$ (where h_i is extended to E by setting it equal to 0 in $E \setminus E_i$) is decreasing and, by Lemma 1.33 in [11], $\nabla h \in L^p(E)$ and ∇h_i converges to ∇h weakly in $L^p(E)$. □

Remark 9. *Suppose that the end E is p-parabolic. Then, the same construction works but, in this case, by the boundary maximum principle characterization of parabolicity, we have $h \equiv 1$.*

2. Sobolev inequalities, volume and hyperbolicity of ends

In this section we show that the validity of an $L^{q,p}$-Sobolev inequality implies that each of the ends of the underlying manifold has infinite volume and is p-hyperbolic.

Theorem 10. *Every end of a complete Riemannian manifold $(M, \langle , \rangle)$ supporting the $L^{q,p}$-Sobolev inequality (1) for some $q > p \ge 1$ is p-hyperbolic and, in particular, has infinite volume.*

Corollary 11. *Suppose that the complete manifold M has (at least) one p-parabolic end. Then the $L^{q,p}$-Sobolev inequality (1) fails.*

The result of Theorem 10 consists of two parts. For the sake of clarity, we treat each part separately.
Volume of ends supporting an $L^{q,p}$-Sobolev inequality. It is elementary to show that if an $L^{q,p}$-Sobolev inequality holds on a manifold then the manifold has infinite volume. Indeed, having fixed x_o in M we consider a family $\{\varphi_R\}_{R>0}$ of cut-off functions satisfying: (a) $0 \le \varphi_R \le 1$; (b) $\varphi_R = 1$ on $B_{R/2}(x_o)$; (c) $\mathrm{supp}(\varphi_R) \subset B_R(x_o)$; (d) $|\nabla \varphi_R| \le 4/R$ on M. Using φ_R into the Sobolev inequality gives

$$S_{q,p}\mathrm{vol}\left(B_{R/2}(x_o)\right)^{1/q} \le S_{q,p}\|\varphi_R\|_{L^q} \le \|\nabla \varphi_R\|_{L^p} \le \frac{4}{R}\mathrm{vol}\left(B_R(o)\right)^{1/p},$$

which, in turn, implies the non-uniform estimate

$$\mathrm{vol}\left(B_R(o)\right) \ge CR^p,$$

for every $R \ge 1$ and for some constant $C = \left(4^{-1}S_{q,p}\mathrm{vol}\left(B_1(o)\right)^{1/q}\right)^p > 0$. In particular $\mathrm{vol}(M) = +\infty$ and at least one of the ends of M has infinite volume.

In order to extend this conclusion to each of the ends we can use a uniform volume estimate due to Carron, [2] (see also [8]) which we state in the form suitable for our purposes.

Proposition 12. *Let E be an an end of the complete manifold M with respect to the compact set K and assume that the $L^{q,p}$-Sobolev inequality (1) holds on E, for some $q > p \ge 1$. Then there exist positive constants C_1*

and C_2 depending only on p, q and $S_{q,p}$ such that, for every geodesic ball $B_R(x_0) \subset E$

$$vol\left(B_R\left(x_0\right)\right) \geq C_1 R^{C_2}. \tag{3}$$

In particular, if $K \subset B_{R_0}(o)$, then for every $x_0 \in E$ with $d(x_0, o) \geq R + R_0$ the ball $B_R(x_0) \subset E$, and E has infinite volume.

Proof. For every $\Omega \subset E$ let

$$\lambda\left(\Omega\right) = \inf \frac{\int_\Omega |\nabla\varphi|^p}{\int_\Omega |\varphi|^p},$$

the infimum being taken with respect to all $\varphi \in W_c^{1,p}\left(\Omega\right)$, $\varphi \not\equiv 0$. By the Sobolev and Hölder inequalities, for every such φ we have

$$\int_\Omega \varphi^p \leq \mathrm{vol}(\Omega)^{\frac{q-p}{q}} \left(\int_\Omega \varphi^q\right)^{\frac{p}{q}} \leq \left(S_{q,p}||\nabla\varphi||_p\right)^p,$$

and therefore

$$vol\left(\Omega\right)^{\frac{q-p}{q}} \lambda\left(\Omega\right) \geq S_{q,p}^p. \tag{4}$$

On the other hand, choosing $\Omega = B_R\left(x_0\right)$ and

$$\varphi\left(x\right) = R - d\left(x, x_0\right)$$

we deduce that

$$\begin{aligned} \lambda\left(B_R\left(x_0\right)\right) &\leq \frac{vol\left(B_R\left(x_0\right)\right)}{\int_{B_R(x_0)} \left(R - d\left(x, x_0\right)\right)^p} \\ &\leq \frac{vol\left(B_R\left(x_0\right)\right)}{\int_{B_{R/2}(x_0)} \left(R - d\left(x, x_0\right)\right)^p} \\ &\leq \frac{2^p vol\left(B_R\left(x_0\right)\right)}{R^p vol\left(B_{R/2}\left(x_0\right)\right)}. \end{aligned} \tag{5}$$

Combining (4) and (5) we obtain

$$vol\left(B_R\left(x_0\right)\right)^{1+\frac{q-p}{q}} \geq 2^{-p} S_{q,p}^p R^p vol\left(B_{R/2}\left(x_0\right)\right),$$

i.e.,

$$vol\left(B_R\left(x_0\right)\right) \geq \left(2^{-p} S_{q,p}^p R^p\right)^\alpha vol\left(B_{R/2}\left(x_0\right)\right)^\alpha,$$

with

$$0 < \alpha = \frac{1}{1 + \frac{q-p}{q}} < 1.$$

Iterating this inequality k-times yields

$$vol\left(B_R\left(x_0\right)\right) \geq 2^{-p\alpha \sum_{j=1}^k j\alpha^j} \left(2^{-p} S_{q,p}^p R^p\right)^{\sum_{j=1}^k \alpha^j} vol\left(B_{R/2^k}\left(x_0\right)\right)^{\alpha^k}.$$

Since

$$vol B_r\left(x_0\right) \sim \omega_0 r^m \text{ as } r \to 0 \;\; (m = \dim M),$$

for k large enough

$$vol\left(B_{R/2^k}\left(x_0\right)\right)^{\alpha^k} \geq \left(\frac{1}{2}\omega_0 R^m 2^{-km}\right)^{\alpha^k}$$

and letting $k \to +\infty$ finally gives

$$vol\left(B_R\left(x_0\right)\right) \geq 2^{-p\bar{\alpha}}\left(2^{-p}S_{q,p}^p R^p\right)^{\frac{\alpha}{1-\alpha}},$$

where

$$\bar{\alpha} = \sum_{j=1}^{+\infty} j\alpha^j,$$

and estimate (3) holds.

To prove the second statement, assume that $x_0 \in E$ is such that $d(x_0, o) \geq R + R_0$, and consider the geodesic ball $B_R(x_0)$. If $x \in \overline{B_{R_0}(o)}$, then by the triangle inequality,

$$d\left(x_0, x\right) \geq d\left(x_0, o\right) - d\left(o, x\right) \geq R,$$

proving that $B_R\left(\bar{x}\right) \cap \overline{B_{R_0}\left(o\right)} = \emptyset$. On the other hand, if E' is a second connected component of $M \backslash K$ and $x'' \setminus B_{R_0}(o)$, let σ be a minimizing geodesic from x_0 to x'. By continuity, σ must intersect $\partial B_{R_0}\left(o\right)$ at some point x_1 and

$$d\left(x_0, x'\right) = \ell\left(\sigma\right) = d(x', x_1) + d(x_1, x_0) > d(x_1, \bar{x}) \geq R.$$

Therefore $B_R\left(x_0\right) \cap E' = \emptyset$ and we conclude that $B_R(x_0) \subset E$. Since $x_0 \in E$ can be chosen in such a way that $d\left(x_0, o\right)$ is arbitrarily large, letting $E \ni x_0 \to \infty$ gives that $vol\left(E\right) = +\infty$. □

Hyperbolicity of ends. The second part of Theorem 10 states that every end of M is p-hyperbolic. In the special case $p = 2$ this conclusion is (essentially) due to Cao-Shen-Zhu, [4], and Li-Wang, [14]; see also [16]. Our proof, which we seems to be conceptually clearer even in the case $p = 2$, is essentially based on the observation that if the $L^{q,p}$ Sobolev inequality (1) holds on M then M is necessarily p-hyperbolic. Indeed, if Ω is any compact domain then, for every $\varphi \in C_c^\infty\left(M\right)$ satisfying $\varphi \geq 1$ on Ω it holds

$$S_{q,p}\mathrm{vol}\left(\Omega\right)^{1/q} \leq S_{q,p}\left\|\varphi\right\|_{L^q} \leq \left\|\nabla\varphi\right\|_{L^p},$$

proving that

$$\mathrm{cap}_p\left(\Omega\right) \geq S_{q,p}^p \mathrm{vol}\left(\Omega\right)^{p/q} > 0.$$

This shows that M is p-hyperbolic, and therefore at least one of its ends is p-hyperbolic. To extend the conclusion to each end E of M, we are naturally led to apply the reasonings to the double $\mathcal{D}\left(E\right)$. By the very definition of the double of a manifold, it turns out that $\mathcal{D}\left(E\right)$ supports the Sobolev inequality (1) outside a compact neighborhood of the glued boundaries. Accordingly, to conclude that E is p-hyperbolic we can make a direct use the following very general theorem that extends to any $L^{q,p}$-Sobolev inequality previous work by Carron, [1].

Theorem 13. *Lat $(M, \langle , \rangle)$ be a possibly incomplete Riemannian manifold. Assume that M has infinite volume and supports the $L^{q,p}$-Sobolev inequality (1) off a compact set $K \subset M$. Then, M is p-hyperbolic and the same Sobolev inequality, possibly with a different constant, holds on all of M.*

Remark 14. *Clearly, if M is complete, according to Proposition 12 the assumption that M has infinite volume is automatically satisfied.*

Proof. Let Ω be a precompact domain with smooth boundary such that $K \subset\subset \Omega$. Let also $W_\varepsilon \approx \partial\Omega \times (-\varepsilon, \varepsilon)$ be a bicollar neighborhood of $\partial\Omega$ such that $W_\varepsilon \subset M \backslash K$, and let $\Omega_\varepsilon = \Omega \cup W_\varepsilon$ and $M_\varepsilon = M \backslash \Omega_\varepsilon$. Note that, by assumption, the $L^{q,p}$-Sobolev inequality with Sobolev constant $S > 0$ holds on M_ε. Furthermore, the same $L^{q,p}$-Sobolev inequality, with some constant $S_\varepsilon > 0$ holds on the compact manifold with boundary $\overline{\Omega_\varepsilon}$ (start with the Euclidean L^1 Sobolev inequality and use Holder inequality a number of times). Now, let $\rho \in C_c^\infty(M)$ be a cut-off function satisfying $0 \leq \rho \leq 1$, $\rho = 1$ on $\Omega_{\varepsilon/2}$ and $\rho = 0$ on M_ε. Next, for any $v \in C_c^\infty(M)$, write $v = \rho v + (1-\rho) v$, and note that $\rho v \in C_c^\infty(\Omega_\varepsilon)$ whereas $(1-\rho) v \in C_c^\infty\left(M_{\varepsilon/2}\right)$ Therefore, we can apply the respective Sobolev inequalities and get

$$\begin{aligned}
\|v\|_{L^q(M)} &\leq \|v\rho\|_{L^q(\Omega_\varepsilon)} + \|v(1-\rho)\|_{L^q\left(M_{\varepsilon/2}\right)} \\
&\leq S_\varepsilon^{-1} \|\nabla(v\rho)\|_{L^p(\Omega_\varepsilon)} + S^{-1} \|\nabla(v(1-\rho))\|_{L^p\left(M_{\varepsilon/2}\right)} \\
&\leq \left(S_\varepsilon^{-1} + S^{-1}\right) \|\nabla v\|_{L^p(M)} + S_\varepsilon^{-1} \|v\nabla\rho\|_{L^p\left(\Omega_\varepsilon \backslash \Omega_{\varepsilon/2}\right)} + S^{-1} \|v\nabla\rho\|_{L^p(\Omega_\varepsilon)} \\
&\leq \left(S_\varepsilon^{-1} + S^{-1}\right) \left\{ \|\nabla v\|_{L^p(M)} + C \|v\|_{L^p(\Omega_\varepsilon)} \right\},
\end{aligned}$$

where $C = \max_M |\nabla\rho|$. Summarizing, we have shown that, for every $v \in C_c^\infty(M)$,

$$\|v\|_{L^q(M)} \leq C_1 \left\{ \|\nabla v\|_{L^p(M)} + \|v\|_{L^p(\Omega_\varepsilon)} \right\}, \tag{6}$$

for a suitable constant $C_1 > 0$.

With this preparation, we now prove that M is p-hyperbolic. To this end, using the fact that $\mathrm{vol}(M) = +\infty$, we choose a compact set $\Omega' \supset \Omega_\varepsilon$ satisfying

$$\mathrm{vol}\left(\Omega'\right)^{1/p} \geq (2C_1)^q \,\mathrm{vol}(\Omega_\varepsilon)^{q/p}.$$

Thus, applying (6) with a test function $v \in C_c^\infty(M)$ satisfying $v = 1$ on Ω', we deduce

$$\mathrm{vol}(\Omega_\varepsilon) \leq C_1^{-1} \mathrm{vol}\left(\Omega'\right)^{1/q} - \mathrm{vol}(\Omega_\varepsilon)^{1/p} \leq \|\nabla v\|_{L^p(M)}.$$

It follows that

$$\mathrm{cap}_p\left(\Omega'\right) \geq \mathrm{vol}(\Omega_\varepsilon) > 0,$$

proving that M is p-hyperbolic.

Finally, we show that the Sobolev inequalities on Ω_ε and on M_ε glue together. According to (6) it suffices to prove that there exists a suitable

constant $E = E(\Omega_\varepsilon) > 0$ such that

$$\|v\|_{L^p(\Omega_\varepsilon)} \le E \|\nabla v\|_{L^p(M)},$$

for every $v \in C_c^\infty(M)$. Since M is p-hyperbolic, this latter inequality follows from Theorem 4 (ii). $\square$

We are now in the position to prove the last part of Theorem 10. Thus, let E be an end with smooth boundary of the complete manifold M supporting the $L^{q,p}$-Sobolev inequality (1). We shall prove that the double $\mathcal{D}(E)$ of E is a p-hyperbolic manifold (without boundary). To this purpose, we note that $\mathcal{D}(E)$ has infinite volume because, by the first part of Theorem 10, E itself has infinite volume. Furthermore, E enjoys the Sobolev inequality (1) outside a compact neighborhood of the glued boundaries. Therefore, a direct application of Theorem 13 yields that $\mathcal{D}(E)$ is a p-hyperbolic manifold, as desired.

3. p-harmonic functions with finite p-energy

This section aims to prove the following theorem that extends to the nonlinear setting previous results of the Li-Tam theory, [13]. We are grateful to I. Holopainen who pointed out that Theorem 15 follows from Theorem 4.6 [10] A more precise result is also contained in the paper by S.W. Kim, and Y.H. Lee, [12].

Theorem 15. *Let $(M, \langle,\rangle)$ be a Riemannian manifold with at least two p-hyperbolic ends (with respect to some smooth, compact domain). Then, there exists a non-constant , bounded p-harmonic function $u \in C(M) \cap C^{1,\alpha}_{loc}(M)$ satisfying $|\nabla u| \in L^p(M)$.*

Proof. Let $E_1, ..., E_n$ be the ends of M with respect to the smooth domain $\Omega \subset\subset M$. By assumption, we may suppose that E_1 and E_2 are p-hyperbolic. Let $\{D_t\}_{t\in\mathbb{N}}$ be a smooth exhaustion of M and set $E_{j,t} = E_j \cap D_t$.

For every $t \in \mathbb{N}$, let $u_t \in C^{1,\alpha}_{loc}(D_t) \cap C\left(\overline{D_t}\right)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_p u_t = 0 & \text{on } D_t \\ u_t = 1 & \text{on } E_1 \cap \partial D_t \\ u_t = 0 & \text{on } E_j \cap \partial D_t,\ j \neq 1. \end{cases}$$

Note that, by the strong maximum principle, $0 < u_t < 1$ in D_t. Moreover, as explained in Lemma 8, the sequence $\{u_t\}_{t\in\mathbb{N}}$ converges, locally uniformly, to a p-harmonic function $u \in C(M) \cap C^{1,\alpha}_{loc}(M)$ satisfying $0 \le u \le 1$. Now, for every $j = 1, 2$, let h_j be the p-harmonic function associated to the ends E_j costructed in Lemma 8. Recall that h_j is the (locally uniform) limit of the p-harmonic function $h_{j,t}$ which satisfy $h_{j,t} = 1$ on ∂E_j and $h_{j,t} = 0$ on $E_j \cap \partial D_t$. Define $k_{1,t} = 1 - h_{1,t}$. Then, comparing u_t and $k_{1,t}$ on $E_{1,t}$ yields that $u_t \ge k_{1,t}$ on $E_{1,t}$. On the other hand, comparing u_t and $h_{2,t}$, gives $u_t \le h_{2,t}$ on $E_{2,t}$. Therefore, taking limits as $t \to +\infty$, we deduce that

$u \geq h_1$ on E_1 and $u \leq k_2$ on E_2. From this, using (3) in Lemma 8, we conclude that u is non-constant. We claim that $|\nabla u| \in L^p(M)$. Indeed, for every $j = 1, ..., n$, let $F_{j,t} = E_j \backslash E_{j,t}$. We think of u_t as extended to all of M by $u_t = 1$ on $F_{1,t}$ and $u_t = 0$ on $\cup_{i=2}^n F_{i,t}$. Then, by construction, u_t is the equilibrium potential of the condenser $(F_{1,t}, \cup_{i=2}^n E_{i,t} \cup \Omega \cup E_1)$ and we have

$$\mathrm{cap}_p\left(F_{1,t}, \cup_{i=2}^n E_{i,t} \cup \Omega \cup E_1\right) = \int_M |\nabla u_t|^p .$$

On the other hand, take $k_{1,t}$ and extend it to be one on $F_{1,t}$. Then, $k_{1,t}$ is the equilibrium potential of the condenser $(F_{1,t}, E_1)$ and we have

$$\mathrm{cap}_p\left(F_{1,t}, E_1\right) = \int_M |\nabla k_{1,t}|^p .$$

By the monotonicity properties of the p-capacity, [11], [5], and recalling that $\int_{E_{1,t}} |\nabla k_{1,t}|^p$ is decreasing in t, we deduce

$$\begin{aligned}\int_M |\nabla u_t|^p &= \mathrm{cap}_p\left(F_{1,t}, \cup_{i=2}^n E_{i,t} \cup \Omega \cup E_1\right) \\ &\leq \mathrm{cap}_p\left(F_{1,t}, E_1\right) = \int_M |\nabla k_{1,t}|^p = \int_{E_{1,t}} |\nabla k_{1,t}|^p \leq C,\end{aligned}$$

for some constant $C > 0$ independent of t. Now observe that, for every domain $D \subset\subset M$, $\nabla u_t \to \nabla u$ weakly in $L^p(D)$ and therefore

$$\int_D |\nabla u|^p \leq \liminf_{t \to +\infty} \int_D |\nabla u_t|^p \leq C.$$

Letting $D \nearrow M$ completes the proof. $\square$

4. A Liouville-type result for p-harmonic functions

In the very recent paper [18], the authors prove the following vanishing result for p-harmonic maps with low regularity.

Theorem 16. *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold with Ricci tensor satisfying*

$$^M Ric \geq -q(x),$$

for some continuous function $q(x) \geq 0$. Let $p \geq 2$ and assume that, for some $s \geq p$ and some $H > s^2/4(s-1)$, the bottom of the spectrum of the Schrodinger operator $L_H = -\Delta - Hq(x)$ satisfies

$$\lambda_1^{-L_H}(M) \geq 0.$$

Then, every p-harmonic map $u : M \to N$ of class C^1 into a non-positively curved manifold N is constant, provided its energy density satisfies the integrability condition

$$\int_{B_R} |du|^s = o(R), \text{ as } R \to +\infty.$$

As a consequence, we deduce the following

Corollary 17. *Let $(M,\langle,\rangle)$ be complete Riemanian manifold such that*

$$^{M}Ric \geq -q(x)$$

for some continuous function $q(x)\geq 0$. Let $p\geq 2$ and assume that the Schrodinger operator $L_H=-\Delta - Hq(x)$ satisfies

$$\lambda_1^{L_H}(M)\geq 0$$

for some $H>p^2/4(p-1)$. Then, every p-harmonic function $u:M\to\mathbb{R}$ of class C^1 and with finite p-energy $|\nabla u|\in L^p(M)$ must be constant.

5. Proof of the main theorem

Putting together the results of the previous sections we obtain a proof of Theorem 1. Indeed, Let $\Omega\subset\subset M$ be fixed. Since M is complete and supports an $L^{q,p}$-Sobolev inequality, according to Theorem 10, all the ends of M with respect to Ω are p-hyperbolic. By contradiction, suppose that there are at least two ends. Then, by Theorem 15 there exists a non-constant p-harmonic function $u\in C^1(M)$ such that $|\nabla u|\in L^p(M)$. Now, by the Ricci curvature assumption and the fact that $p\geq 2$, we can apply Corollary 17 to deduce that u is constant. Contradiction.

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